5. Nitsche, J., Über Unstätigkeiten in den Ableitungen von Lösungen quasilinearer hyperbolischer Differential – gleichungsysteme. J. Rational Mech. and Analysis, Vol. 2, Nr. 2, 1953.
6. Rubina, L. I., The propagation of weak discontinuities in the systems of equations of magnetogasdynamics. PMM Vol. 33, №5, 1969.
7. Kamke, E., Bemerkungen zur Theorie der partiellen differentialgleichenden erster Ordnung. Math. Z., 1943, Bd. 49, №2, s. 256 – 284.
8. Wood, R. W., Physical Optics. 3rd ed. N.Y. Macmillan, 1934.

Translated by L. K.

UDC 532.5

ON CERTAIN CLASSES OF QUASI-STATIONARY FLOWS OF A PERFECT INCOMPRESSIBLE FLUID

PMM Vol. 36, №3, 1972, pp. 444-449
N. N. GORBANEV
(Tomsk)
(Received July 5, 1971)

We consider a nonstationary flow with stationary streamlines (i.e. a quasi-stationary flow) of a perfect incompressible fluid in a conservative external force field.

A specific property is obtained for the field of velocity directions of an irrotational quasi-stationary flow, a relationship determined between the moduli of the velocities of the quasi-stationary and the stationary flow with the same streamlines, and a possibility of existence of rotational and irrotational quasi-stationary flows with common streamlines is studied.

In [1 – 3] the necessary conditions are obtained for the field of unit vectors in order that it may serve as a field of velocity directions of stationary flow of an incompressible fluid. An analogous problem for a quasi-stationary flow is solved in [4] only for the case when the field of velocity directions is rectilinear.

1. Let us denote the unit velocity direction vector by $\mathbf{e}$ and the streamline curvature vector by $\mathbf{k}$. The field of vectors $\mathbf{l} = \mathbf{k} - \mathbf{e} \operatorname{div} \mathbf{e}$ is called the field of adjoint vectors of the field of $\mathbf{e}$. A vector field is called holonomic [5], if there exists a family of surfaces orthogonal to the field. The quantity $H = \operatorname{div} \mathbf{e}$ is the mean curvature of the field of $\mathbf{e}$ [6]. A field of mean zero curvature is called the minimal field [1].

We shall now find the necessary and sufficient geometrical conditions for the field of unit vectors $\mathbf{e}$ in order that it may serve as a field of velocity directions of an irrotational quasi-stationary flow.

Theorem 1. The field of unit vectors $\mathbf{e}$ may serve as a field of velocity directions of an irrotational quasi-stationary flow if and only if

1) the field of $\mathbf{e}$ is holonomic ($\mathbf{e} \cdot \operatorname{rot} \mathbf{e} = 0$);

2) the field of its adjoint vectors is potential ($\operatorname{rot} \mathbf{l} = 0$).

Fields of unit vectors satisfying these conditions exist and can be determined to within two functions of two arguments. If the field of directions is specified, then the velocity modulus of an irrotational quasi-stationary flow is determined to within one function of a single argument.

The conditions (1) and (2) turn out to be the characteristic properties of the fields of velocity directions of an irrotational stationary flow. Since specifying the field of velocity directions is equivalent to specifying a congruence of the streamlines, the following theorem is valid.

Theorem 2. A congruence of the lines can serve as a congruence of the streamlines of an irrotational quasi-stationary flow of a perfect incompressible fluid if and only if it can be a congruence of the streamlines of an irrotational stationary flow.

Note. The following relation exists between the velocity moduli of the irrotational quasi-stationary and the irrotational stationary flow with common streamlines: if W is the velocity modulus of the irrotational stationary flow, then $\psi(t)\, W$, where $\psi(t)$ is an arbitrary function of time only, is the velocity modulus of the irrotational quasi-stationary flow with the same streamlines. Conversely, the modulus V of velocity of the irrotational quasi-stationary flow can always be represented in the form $V = \psi^*(t)\, W$, where $\psi^*(t)$ is a certain function of time.

Let us investigate the possibility of existence of both,an irrotational and a rotational quasi-stationary flow with common streamlines.

Theorem 3. The field of unit vectors $\mathbf{e}$ of an irrotational quasi-stationary flow can serve as a field of velocity directions of a rotational quasi-stationary flow if and only if the following conditions hold in addition to the conditions (1) and (2) of Theorem 1:

3) the field of the curvature vectors $\mathbf{k}$ is holonomic, i.e. $\mathbf{k}\cdot \mathrm{rot}\, \mathbf{k} = 0$;

4) $\mathbf{e}\cdot \mathrm{grad}\, H = H^2$.

The results of [1] imply that the field of velocity directions of the irrotational and rotational stationary flows with common streamlines is minimal ($H = 0$). In this case $\mathbf{I} = \mathbf{k}$ and the conditions (3) and (4) are satisfied. It follows that both the irrotational and rotational quasi-stationary flows with common streamlines exist.

Let us denote the velocity moduli of the quasi-stationary and the stationary flow by V^* and W^*, respectively, and by V and W for the particular case of an irrotational quasi-stationary and an irrotational stationary flows. We shall now consider the relation between the moduli of the irrotational and rotational quasi-stationary and stationary flows with common streamlines.

Theorem 4. If the field of velocity directions of an irrotational quasi-stationary flow is minimal, the sum of the modulus V of this flow and the modulus W^* of the stationary flow with the same streamlines is equal to the modulus V^* of velocity of the quasi-stationary flow with the same streamlines, i.e. we have the relation

$$V^* = V + W^*$$

Taking into account the Note in Theorem 2, we can write this relation in the form $V^* = \psi(t) W + W^*$, remembering that if W^* is the velocity modulus of a rotational stationary flow, then V^* is a velocity modulus of a rotational flow. On the other hand, if W^* denotes the velocity modulus of an irrotational stationary flow, then V^* is also a velocity modulus of an irrotational quasi-stationary flow with the same streamlines. Since for a given minimal field of velocity directions of an irrotational stationary flow W^* is determined to within one function of a single argument [1], then

$V^* = \psi(t) W + W^*$ is determined to within two functions of a single argument.

The boundary conditions for a quasi-stationary flow can be specified in accordance with the relation connecting the quasi-stationary and the stationary flow, and the boundary conditions for a stationary motion.

2. The above results have been obtained by applying the methods of the Cartan calculus of exterior forms to the problem of compatibility of the systems of differential equations of hydrodynamics.

Let a non-rectilinear field of unit vectors $\mathbf{e}$ be specified in a three-dimensional region of the Euclidean space, this is equivalent to specifying a congruence of lines. The case of a rectilinear field was dealt with in [4] and is therefore omitted. Let us assign to each point of definition of the given field a Frenet n-hedron for a vector line of this field, passing through this point. Let $\mathbf{e}_3 = \mathbf{e}$, $\mathbf{e}_2$, $\mathbf{e}_1$ be unit vectors of the tangent, the principal normal and the binormal to the line, respectively, and $\mathbf{M}$ be the radius vector of the point. The differentials of the basis vectors $\mathbf{e}_i$ and the differential of the radius vector $\mathbf{M}$ can be expanded at every point over the basis at this point to yield so-called derivative formulas [7]

$$\mathbf{dM} = \omega^i \mathbf{e}_i, \qquad \mathbf{de}_i = \omega_i^{\ j} \mathbf{e}_j \qquad (i, j = 1, 2, 3) \tag{2.1}$$

The matrix $\|\omega_i^{\ j}\|$ is skew symmetric, i.e. $\omega_i^{\ j} = -\omega^i_{\ j}$. Therefore only three of the forms $\omega_i^{\ j}$, namely $\omega_3^{\ 2}$, $\omega_3^{\ 1}$ and $\omega_1^{\ 2}$ are distinct. Since the region is three-dimensional, the forms ω^1, ω^2 and ω^3 are linearly independent and can be used to express all the remaining forms

$$\omega_3^{\ 2} = a_1\omega^1 + a_2\omega^2 + a_3\omega^3, \qquad \omega_3^{\ 1} = b_1\omega^1 + b_2\omega^2 \tag{2.2}$$
$$\omega_1^{\ 2} = c_1\omega^1 + c_2\omega^2 + c_3\omega^3$$

Here $b_3 = 0$ since $\mathbf{e}_2$ is directed along the principal normal to the line. The system (2.1) is completely integrable, then the following structural equations hold:

$$D\omega^i = [\omega^j \omega_j^{\ i}], \qquad D\omega_j^{\ i} = [\omega_j^{\ m} \omega_m^{\ i}] \qquad (i, j, m = 1, 2, 3) \tag{2.3}$$

where $D\omega^i$ is the external differential and $[\omega^j \omega_j^{\ i}]$ is the external product [7].

Let $\mathbf{V} = V\mathbf{e}_3$ denote the velocity of flow of a perfect incompressible fluid. Then the system of equations of hydrodynamics can be written in the total differentials as follows:

$$dV = V_1\omega^1 + V_2\omega^2 - VH\omega^3 + V_t dt$$
$$d\varphi = a_3V^2\omega^2 + (V_t - HV^2)\,\omega^3 + \varphi_t\, dt \tag{2.4}$$

The first equation is the continuity equation and the second is the Euler equation in which φ denotes the acceleration potential. Since

$$\operatorname{rot} \mathbf{V} = (V_2 - a_3V)\,\mathbf{e}_1 - V_1\mathbf{e}_2 + V(a_1 - b_2)\,\mathbf{e}_3,$$

we have for the irrotational flow

$$V_1 = 0, \qquad V_2 = a_3V, \qquad a_1 - b_2 = 0 \tag{2.5}$$

The third condition means that the field of velocity directions of the irrotational quasi-stationary flow is a holonomic field [5]. As we consider here only the flows which have streamlines common with the irrotational quasi-stationary flow, we shall limit our considerations to the holonomic unit vector fields.

A system defining an irrotational flow has the form

$$dV = a_3 V\omega^2 - HV\omega^3 + V_t dt,\ d\varphi = a_3 V^2\omega^2 + (V_t - HV^2)\,\omega^3 + \varphi_t dt \tag{2.6}$$

We shall clarify now which unit vector fields may serve as the fields of velocity directions for an irrotational quasi-stationary flow, i.e. what conditions must be imposed on the coefficients a_i, b_i and c_i (except $a_1 - b_2 = 0$) of (2.2), for the system (2.6) to be in involution [7]. Performing the external differentiation on (2.6), we obtain the following system of external bilinear differential equations:

$$[dV_t dt] = [\omega^1\omega^3]\,(H_1 + a_3 y)\,V + [\omega^2\omega^3]\,(H_2 + a_{33} - b_1 a_3)\,V + [\omega^2 dt]\,a_3 V_t - [\omega^3 dt]\,HV_t$$

$$[dV_t\omega^3] + [d\varphi_t dt] = [\omega^1\omega^3]\,(H_1 + a_3 y)\,V^2 + [\omega^2\omega^3]\,\{(H_2 + a_{33} - b_1 a_3)\,V^2 + a_3 V_t\} + [\omega^2 dt]\,2a_3 VV_t + [\omega^3 dt]\,(-2HVV_t) \tag{2.7}$$

where $y = a_1 - c_3$, and H_1, H_2 and a_{33} are given by

$$dH = H_1\omega^1 + H_2\omega^2 + H_3\omega^3,\quad da_3 = a_{31}\omega^1 + a_{32}\omega^2 + a_{33}\omega^3$$

In order that the field of unit vectors may serve as a field of velocity directions of irrotational quasi-stationary flow, it is necessary and sufficient to satisfy the following conditions:

$$a_1 - b_2 = 0,\qquad H_1 + a_3 y = 0,\qquad H_2 + a_{33} - b_1 a_3 = 0 \tag{2.8}$$

Indeed the conditions (2.8) are necessary for a nontrivial ($V \neq 0$) solution of (2.7) to exist. Conversely, if these conditions hold, the system (2.7) and therefore (2.6) will, after the substitution $\omega^2 = \omega_0^2 + dt$, be in involution relative to the unknowns V, φ, V_t and φ_t.

The first condition of (2.8) represents the condition for the field $\mathbf{e}_3$ to be holonomic and the remaining two, with the first one taken into account, mean that the field of adjoint vectors $\mathbf{l}$ of the field of $\mathbf{e}_3$ is potential.

Let us find the degree of admissible arbitrariness under which the vector fields satisfying these conditions may exist. The first and third condition of (2.8) together with the structure equations imply that $2a_1c_2 + a_{11} + a_{22} + a_{33} - b_1 a_3 = a_2 c_1 - b_1 c_1$. Continuing the system (2.2) and taking these conditions into account, we obtain

$$\omega_1^2 = c_1\omega^1 + c_2\omega^2 + c_3\omega^3$$

$$db_1 = (-a_3 y - a_{21})\omega^1 + (a_3 b_1 - a_{33} - a_{22})\omega^2 + (a_1 c_3 - a_3 c_1 - a_1 y - b_1^2)\omega^3$$

$$da_1 = (a_3 b_1 + a_2 c_1 - b_1 c_1 - 2a_1 c_2 - a_{22} - a_{33})\omega^1 + (a_{21} + 2a_1 c_1 + a_2 c_2 - b_1 c_2)\,\omega^2 + (a_2 c_3 - c_2 a_3 - a_1 b_1 - a_1 a_2 - b_1 c_3)\omega^3 \tag{2.9}$$

$$da_2 = a_{21}\omega^1 + a_{22}\omega^2 + a_{23}\omega^3,\qquad da_3 = -c_2 a_2\omega^1 + (a_{23} + a_1^2 + a_2^2 + a_3^2 + 2a_1 c_3)\omega^2 + a_{33}\omega^3$$

Performing now the external differentiation on (2.9) we obtain a bilinear system in involution. Its solution exists and can be determined to within two functions of two arguments. This can be seen when constructing a regular chain of solutions using the Kähler's method [7].

If a unit vector field satisfying the conditions (2.8) is specified, then a solution of (2.7) exists and can be determined to within two functions of a single argument. This can be verified by making the substitution $\omega^2 = \omega_0^2 + dt$ and constructing a regular chain of solutions using the Kähler's method, whereupon V is determined to within one function of a single argument. This proves Theorem 1.

We shall next prove Theorem 2. The system determining an irrotational ($W_1 = 0$, $W_2 = a_3 W$, $a_1 - b_2 = 0$) stationary flow with the velocity modulus equal to W, has the form

$$dW = a_3 W\omega^2 - HW\omega^3, \qquad d\varphi = a_3 W^2\omega^2 - HW^2\omega^3 \tag{2.10}$$

This is equivalent to the system

$$W^{-1}dW = a_3\omega^2 - H\omega^3, \qquad \varphi = {}^1/_2\, W^2 + \text{const} \tag{2.11}$$

The system (2.11) is fully integrable if rot $(a_3\mathbf{e}_2 - H\mathbf{e}_3) = \text{rot} I = 0$. Comparison with the conditions of Theorem 1 completes the proof of Theorem 2.

The system (2.6) implies that

$$(\ln V)_{t1} = (\ln V)_{t2} = (\ln V)_{t3} = 0$$

Consequently $(\ln V)_t = \Phi(t)$. Integrating this relation and taking antilogarithms we find that $V = \psi(t)\, W$, where $W_t = 0$. From $(\ln V)_1 = 0$, $(\ln V)_2 = a_3$, $(\ln V)_3 = -H$ and $W_t = 0$ it follows that $W_1 = 0$, $W_2 = a_3 W$, $W_3 = -HW$, $W_t = 0$, i.e. W satisfies system (2.10). Conversely, if W satisfies (2.10), then $\psi(t) W$, where $\psi(t)$ is an arbitrary function, satisfies (2.6) (obviously, they have different acceleration potentials). This proves the validity of the Note in Theorem 2.

Let the field of velocity directions of an irrotational quasi-stationary flow, i.e. the field of unit vectors satisfying the corresponding conditions (2.8), be given. We shall investigate under what conditions this field may serve as the field of velocity directions of a rotational quasi-stationary flow. The system defining the quasi-stationary flow in this case has the form

$$dV = V_2\omega^2 - HV\omega^3 + V_t dt, \quad d\varphi = a_3 V^2\omega^2 + (V_t - HV^2)\,\omega^3 + \varphi_t dt \tag{2.12}$$

External differentiation yields the following system:

$$\begin{aligned}
&[dV_2\omega^2] + [dV_t dt] = -c_2 V_2 [\omega^1\omega^2] - (H_1 V + yV_2)[\omega^3\omega^1] + \\
&+ [\omega^2\omega^3](H_2 V - Ha_3 V + HV_2 + a_2 V_2) - [\omega^3 dt] HV_t \\
&[dV_t\omega^3] + [d\varphi_t dt] = [\omega^2\omega^3](a_3 V_t - 2Ha_3 V^2 + \\
&+ 2HVV_2) + 2a_3 VV_t [\omega^2 dt] - 2HVV_t [\omega^3 dt]
\end{aligned} \tag{2.13}$$

from which we find that $H_1 V + yV_2 = 0$, but $H_1 + a_3 y = 0$, therefore $y\,(V_2 - a_3 V) = 0$. If $y = a_1 - c_3 \neq 0$, then $V_2 = a_3 V$, i.e. if the field of curvature vectors $\mathbf{k}$ is nonholonomic, no rotational stationary flow exists possessing the same streamlines as the given irrotational flow. If $y = 0$, the system (2.13) is not in involution. Continuation and external differentiation yields

$$\begin{aligned}
&[dV_{22}\omega^2] = A\,[\omega^1\omega^2] + B\,[\omega^2\omega^3] + E\,[\omega^2 dt] + [\omega^3 dt] 2V\,(V_2 - a_3 V)(H^2 - H_3) \\
&[dV_{tt} dt] = [\omega^2\omega^3] 2V(V_2 - a_3 V)(H_3 - H^2) + F\,[\omega^2 dt] + G\,[\omega^3 dt] + K\,[\omega^1 dt] \\
&[dV_{tt}\omega^3] + [d\varphi_{tt} dt] = L\,[\omega^2\omega^3] + P\,[\omega^3\omega^1] + N\,[\omega^2 dt] + Q\,[\omega^3 dt]
\end{aligned} \tag{2.14}$$

The form of the coefficients $A, B, E, F, G, K, L, P, N$ and Q is not important as long as they remain different from zero. From (2.14) it follows that $2V\,(V_2 - a_3 V)(H^2 - H_3) = 0$, and the condition $H_3 = H^2$ is necessary for a rotational flow to exist. The conditions $H_3 = H^2$ and $y = 0$ are also sufficient for the reason that, provided that the field of velocity directions without a rotational motion satisfies these conditions, the system (2.14) is in involution; its solution exists and is determined to within three functions of a single argument and V is determined to within two functions of a single argument. This can be verified by making the substitution $\omega^2 = \omega_0^2 + dt$ and constructing a regular chain of solutions according to the Kähler's method. The condition $y = a_1 - c_3 = 0$

means that the field of curvature vectors $\mathbf{k} = a_3\mathbf{e}_2$ of the given field of $\mathbf{e}_3$ is holonomic [5]. This completes the proof of Theorem 3.

Suppose that the congruence of the streamlines of the irrotational stationary flow is minimal, i.e. that the conditions $a_1 - b_2 = 0$, $H = 0$, $y = 0$ and $a_{33} - b_1a_3 = 0$ are satisfied. Then both rotational and irrotational flows, in either stationary or quasi-stationary state, with these streamlines exist. In this case the system (2.12) defining a quasi-stationary flow has the form

$$dV^* = V_2^*\omega^2 + V_t^*dt, \quad d\varphi^* = a_3(V^*)^2\omega^2 + V_t^*\omega^3 + \varphi_t^*dt \tag{2.15}$$

and the stationary flow is defined by

$$dW^* = W_2^*\omega^2, \qquad d\varphi_1 = a_3(W^*)^2\ \omega^2 \tag{2.16}$$

If $W_2^* = a_3W^*$, the flow is irrotational, i.e. the system

$$dW = a_3W\omega^2, \qquad d\varphi = a_3W^2\omega^2 \tag{2.17}$$

defines an irrotational stationary flow.

From (2.15) it follows that $(\ln V_t^*)_{t1} = (\ln V_t^*)_{t2} = (\ln V_t^*)_{t3} = 0$. Therefore $(\ln V_t^*)_t = \Phi(t)$. Integrating we obtain $V^* = \psi(t)W + W^*$, where $W_t = W_t^* = 0$. Since $V_{t1}^* = V_{t3}^* = 0$, and $V_{t2}^* = a_3V_t^*$, then $W_1 = W_3 = 0$ and $W_2 = a_3W$, therefore W satisfies (2.17). From $V_1^* = V_3^* = 0$ and $W_1 = W_3 = 0$ follows $W_1^* = W_3^* = 0$, i.e. W^* satisfies (2.16).

Conversely, let W satisfy (2.17) and W^* satisfy (2.16), and let $\psi(t)$ be an arbitrary function of t. Making in (2.15) the substitution $V^* = \psi(t)W + W^*$ and performing external differentiation with (2.16) and (2.17) taken into account, we obtain $[d\varphi_t^*dt] = L[\omega^2dt] + K[\omega^3dt]$, where L and K need not be known exactly. This equation is in involution, the function φ^* is determined to within one function of a single argument provided that W, W^* and $\psi(t)$ are specified. In other words, $\psi(t)W + W^*$ may serve as the velocity modulus for a quasi-stationary flow with specified streamlines. This proves Theorem 4.

3. Examples of the line congruences which may serve as congruences of the streamlines of both rotational and irrotational flows, in either quasi-stationary or stationary state, can be obtained by adding $a_2 = 0$ to the conditions $a_1 - b_2 = 0$, $a_1 - c_3 = 0$, $H = 0$ and $a_{33} - b_1a_3 = 0$. Then $c_2 = 0$ and ω^2 becomes a total differential. Setting $\omega^2 = dU$ at $a_1 = 0$ we obtain a congruence of circles lying on coaxial rectilinear circular cylinders. The motion is plane-parallel and the flow velocity modulus $V = \psi(t)r^{-1} + V_0(r)$, where r is the radius of the cylinder and $V_0(r)$ is an arbitrary function of r. For $a_1 \neq 0$ the streamlines become helical lines lying on coaxial, rectilinear circular cylinders, and their pitch $h = Q^2 + r^2$, where r is the radius of the relevant cylinder and Q is a nonzero constant. In this case $V = \psi(t)r(1 + Q^2r^2)^{-1/2} + V_0(r)$. The coaxial helices are orthogonal to a minimal helicoid with the same axis for any fixed value of Q. Such examples were considered in [8] for a stationary flow.

BIBLIOGRAPHY

1. Biushgens, S.S., Geometry of a stationary flow of a perfect incompressible fluid. Izv. Akad. Nauk SSSR, Ser. Matem., Vol. 12, 1948.
2. Biushgens, S.S., On streamlines I. Dokl. Akad. Nauk SSSR, Vol. 78, №4, 1951.
3. Biushgens, S.S., On streamlines II. Dokl. Akad. Nauk SSSR, Vol. 84, №5, 1952.
4. Biushgens, S.S., Geometry of an unsteady flow of a perfect incompressible

fluid., Izv. Akad. Nauk SSSR, Ser. matem., Vol. 24, 1960.
5. Slukhaev, V. V., On geometrical theory of the stationary motion of a fluid. Dokl. Akad. Nauk SSSR, Vol. 196, №3, 1971.
6. Biushgens, S. S., Vector field geometry. Izv. Akad. Nauk SSSR, Ser. matem., Vol. 10, 1946.
7. Finikov, S. P., Cartan's Method of Exterior Forms in Differential Geometry. M., Gostekhizdat, 1948.
8. Hamel, G., Potentialstromungen mit konstanter Geschwindigket, Sitzber. press. Akad. Wiss. Physik. Math. Kl., Bd. 146, s. 5–20, 1937.

Translated by L. K.